Generalized Integral Operators and Schwartz Kernel Theorem

A. Delcroix
Equipe Analyse Algébrique Non Linéaire
Laboratoire Analyse, Optimisation, Contrôle
Faculté des sciences - Université des Antilles et de la Guyane
97159 Pointe-à-Pitre Cedex Guadeloupe

February 1, 2008

Abstract

In connection with the classical Schwartz kernel theorem, we show that in the framework of Colombeau generalized functions a large class of linear mappings admit integral kernels. To do this, we need to introduce news spaces of generalized functions with slow growth and the corresponding adapted linear mappings. Finally, we show that in some sense Schwartz' result is contained in our main theorem.

Contents

Mathematics Subject Classification (2000): 45P05, 46F05, 46F30, 47G10

Keywords: Schwartz kernel theorem, generalized functions, integral operator, distributions.

1 Introduction

It is well known that the framework of Schwartz distributions is not suitable for setting and solving many differential or integral problems with singular coefficients or data. A natural approach to overcome this difficulty consists in replacing the given problem by a one-parameter family of smooth problems. This is done in most theories of generalized functions and, for example, in Colombeau simplified theory which we are going to use in the sequel. (For details, see the monographies [2], [7], [12] and the references therein.)

In this paper, we continue the investigations in the field of generalized integral operators initiated by the pioneering work of D. Scarpalezos [16], and carried on by J.-F. Colombeau (personal communications and [1]) in view of applications to physics and by C. Garetto *et alii* ([6]) with applications to pseudo differential operators theory and questions of regularity.

More precisely, the following results holds: Every H belonging to $\mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$ defines a linear operator from $\mathcal{G}_C(\mathbb{R}^n)$ to $\mathcal{G}(\mathbb{R}^m)$ by the formula

$$\widetilde{H} : \mathcal{G}_C(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^m), \quad f \mapsto \widetilde{H}(f) \text{ with } \widetilde{H}(f)(x) = \left[\int H_\varepsilon(x,y) f_\varepsilon(y)\,\mathrm{d}y\right],$$

where $(H_\varepsilon)_\varepsilon$ (*resp.* $(f_\varepsilon)_\varepsilon$) is any representative of H (*resp.* f) and $[\,\cdot\,]$ is the class of an element in $\mathcal{G}(\mathbb{R}^d)$. ($\mathcal{G}(\mathbb{R}^d)$ denotes the usual quotient space of Colombeau simplified generalized functions, while $\mathcal{G}_C(\mathbb{R}^d)$ is the subspace of elements of $\mathcal{G}(\mathbb{R}^d)$ compactly supported: See section 2 for the mathematical framework.)

Conversely, in the distributional case, the well known Schwartz kernel theorem asserts that each linear map Λ from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^m)$ continuous for the strong topology of $\mathcal{D}'$ can be represented by a kernel $K \in \mathcal{D}'(\mathbb{R}^m \times \mathbb{R}^n)$ that is

$$\forall f \in \mathcal{D}(\mathbb{R}^n), \ \ \forall \varphi \in \mathcal{D}(\mathbb{R}^m), \ \ (\Lambda(f), \varphi) = (K, \varphi \otimes f).$$

Let us recall here that $\mathcal{D}(\mathbb{R}^n)$ is embedded in $\mathcal{G}_C(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^m)$ in $\mathcal{G}(\mathbb{R}^m)$: In the spirit of Schwartz theorem, we prove that in the framework of Colombeau generalized functions any net of linear maps $(L_\varepsilon : \mathcal{D}(\mathbb{R}^n) \to \mathrm{C}^\infty(\mathbb{R}^m))_\varepsilon$ satisfying some growth property with respect to the parameter ε (the *strongly moderate nets*) gives rise to a linear map $L : \mathcal{G}_C(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^m)$ which can be represented as an integral operator. This means that there exists a generalized function $H_L \in \mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$ such that

$$L(f) = \int H_L(\cdot, y) f(y)\,\mathrm{d}y$$

for any f belonging to a convenient subspace of $\mathcal{G}_C(\mathbb{R}^n)$.

Moreover, this result is strongly related to Schwartz Kernel theorem in the following sense. We can associate to each linear operator $\Lambda : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^m)$ satisfying the hypothesis above mentioned a strongly moderate map L_Λ and consequently a kernel $H_{L_\Lambda} \in \mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$ with the following equality property: For all f in $\mathcal{D}(\mathbb{R}^n)$, $\Lambda(f)$ and $\widetilde{H}_{L_\Lambda}(f)$ are equal in the generalized distribution sense [15] that is, for all $k \in \mathbb{N}$ and $\left(H_{L_{\Lambda,\varepsilon}}\right)_\varepsilon$ representative of H_{L_Λ},

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^m), \ \ \langle \Lambda(f), \Phi \rangle - \int \left(\int H_{L_{\Lambda,\varepsilon}}(x,y) f(y)\,\mathrm{d}y\right) \varphi(x)\,\mathrm{d}x = \mathrm{O}\left(\varepsilon^k\right), \text{ for } \varepsilon \to 0.$$

The paper can be divided in two parts. The first part, formed by section 2 and section 3, introduces all the material which is needed in the sequel. We mention here in particular the notion of *spaces of generalized functions with slow growth*, which are subspaces of the usual space $\mathcal{G}(\mathbb{R}^d)$ with additional limited growth property with respect to the parameter ε. Lemma 16 shows one feature of those spaces (used for the proof of the main results): Convolution admits on them as unit some special δ-nets, whereas with result is false in $\mathcal{G}(\mathbb{R}^d)$. The second part, formed by the two last sections, is devoted to the definition of strongly moderate nets, the statement of the main results and their proofs.

2 Colombeau type algebras

2.1 The sheaf of Colombeau simplified algebras

Let C^∞ be the sheaf of complex valued smooth functions on $\mathbb{R}^d$ ($d \in \mathbb{N}$) with the usual topology of uniform convergence. For every open set Ω of $\mathbb{R}^d$, this topology can be described by the family of semi norms

$$p_{K,l}(f) = \sup_{|\alpha| \leq l, K \Subset \Omega} |\partial^\alpha f(x)|$$

where the notation $K \Subset \Omega$ means that the set K is a compact set included in Ω.

Let us set

$$\mathcal{F}(\mathrm{C}^\infty(\Omega)) = \left\{ (f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \mid \forall l \in \mathbb{N},\ \forall K \Subset \Omega,\ \exists q \in \mathbb{N},\ p_{K,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{-q}) \text{ for } \varepsilon \to 0 \right\},$$

$$\mathcal{N}(\mathrm{C}^\infty(\Omega)) = \left\{ (f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \mid \forall l \in \mathbb{N},\ \forall K \Subset \Omega,\ \forall p \in \mathbb{N},\ p_{K,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{p}) \text{ for } \varepsilon \to 0 \right\}.$$

Lemma 1 *[10] and [11]*
i. The functor $\mathcal{F} : \Omega \to \mathcal{F}(\mathrm{C}^\infty(\Omega))$ defines a sheaf of subalgebras of the sheaf $(\mathrm{C}^\infty)^{(0,1]}$
ii. The functor $\mathcal{N} : \Omega \to \mathcal{N}(\mathrm{C}^\infty(\Omega))$ defines a sheaf of ideals of the sheaf $\mathcal{F}$.

We shall note prove in detail this lemma but quote the two mains arguments:
i. For each open subset Ω of X, the family of seminorms $(p_{K,l})$ related to Ω is compatible with the algebraic structure of $\mathcal{E}(\Omega)$; In particular:

$$\forall l \in \mathbb{N},\ \forall K \Subset \Omega,\ \ \exists C \in \mathbb{R}_+^*,\ \ \forall (f,g) \in (\mathrm{C}^\infty(\Omega))^2\ \ p_{K,l}(fg) \leq C p_{K,l}(f)\, p_{K,l}(g),$$

ii. For two open subsets $\Omega_1 \subset \Omega_2$ of $\mathbb{R}^d$, the family of seminorms $(p_{K,l})$ related to Ω_1 is included in the family of seminorms related to Ω_2 and

$$\forall l \in \mathbb{N},\ \forall K \Subset \Omega_1,\ \ \forall f \in \mathrm{C}^\infty(\Omega_2),\ \ p_{K,l}\left(f_{|\Omega_1}\right) = p_{K,l}(f).$$

Definition 2 *The sheaf of factor algebras*

$$\mathcal{G} = \mathcal{F}(\mathrm{C}^\infty(\cdot)) / \mathcal{N}(\mathrm{C}^\infty(\cdot))$$

is called the sheaf of Colombeau type algebras.

The sheaf $\mathcal{G}$ turns to be a sheaf of differential algebras and a sheaf of modulus on the factor ring $\overline{\mathbb{C}} = \mathcal{F}(\mathbb{C}) / \mathcal{N}(\mathbb{C})$ with

$$\mathcal{F}(\mathbb{K}) = \left\{ (r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \mid \exists q \in \mathbb{N},\ |r_\varepsilon| = \mathrm{O}\left(\varepsilon^{-q}\right) \text{ for } \varepsilon \to 0 \right\},$$
$$\mathcal{N}(\mathbb{K}) = \left\{ (r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \mid \forall p \in \mathbb{N},\ |r_\varepsilon| = \mathrm{O}(\varepsilon^{p}) \text{ for } \varepsilon \to 0 \right\},$$

with $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R},\ \mathbb{R}_+$.

Notation 3 *In the sequel we shall note, as usual, $\mathcal{G}(\Omega)$ instead of $\mathcal{G}(\mathrm{C}^\infty(\Omega))$ the algebra of generalized functions on Ω. For $(f_\varepsilon)_\varepsilon \in \mathcal{F}(\mathrm{C}^\infty(\Omega))$, $[(f_\varepsilon)_\varepsilon]$ will be its class in $\mathcal{G}(\Omega)$.*

2.2 Generalized functions with compact supports

Let us mention here some remarks about generalized functions with compact supports, which will be useful in the sequel.

As $\mathcal{G}$ is a sheaf, the notion of support of a section $f \in \mathcal{G}(\Omega)$ make sense. The following definition will be sufficient for this paper.

Definition 4 *The support of a generalized function $f \in \mathcal{G}(\Omega)$ is the complement in Ω of the largest open subset of Ω where f is null.*

Notation 5 *We denote by $\mathcal{G}_C(\Omega)$ the subset of $\mathcal{G}(\Omega)$ of elements with compact supports.*

Lemma 6 *Every $f \in \mathcal{G}_C$ has a representative $(f_\varepsilon)_\varepsilon$, such that each f_ε has the same compact support.*

There is an other way to introduce generalized functions with compact support more natural in the sequel. We start from the algebra $\mathcal{D}(\Omega)$ considered as the inductive limit of

$$\mathcal{D}_j(\Omega) = \mathcal{D}_{K_j}(\Omega) = \{f \in \mathcal{D}(\Omega) \mid \operatorname{supp} f \subset K_j\}$$

where:

i. $(K_j)_{j\in\mathbb{N}}$ is an increasing sequence of relatively compact subsets exhausting Ω, with $K_j \subset \overset{\circ}{K}_{j+1}$;

ii. $\mathcal{D}_j(\Omega)$ is endowed with the family of semi norms

$$p_{j,l}(f) = \sup_{|\alpha|\leq l, x\in K_j} |\partial^\alpha f(x)|.$$

The topology on $\mathcal{D}(\Omega)$ does not depend on the particular choice of the sequence $(K_j)_{j\in\mathbb{N}}$. Construction of spaces of generalized functions based on projective or inductive limits have already been considered (see e.g. [3], [14]). We just recall it briefly here. Let fix $(K_j)_{j\in\mathbb{N}}$ a sequence of compact sets satisfying **i.** and set

$$\begin{aligned} \mathcal{F}(\mathcal{D}(\Omega)) &= \cup_{j\in\mathbb{N}} \mathcal{F}_j(\Omega) \qquad (1)\\ \text{with } \mathcal{F}_j(\Omega) &= \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{D}_j(\Omega)^{(0,1]} \;\middle|\; \forall l \in \mathbb{N},\ \exists q \in \mathbb{N},\ p_{j,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-q}\right) \text{ for } \varepsilon \to 0 \right\} \\ \mathcal{N}(\mathcal{D}(\Omega)) &= \cup_{n\in\mathbb{N}} \mathcal{N}_n(\Omega)\ , \\ \text{with } \mathcal{N}_j(\Omega) &= \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{D}_j(\Omega)^{(0,1]} \mid \forall l \in \mathbb{N},\ \forall p \in \mathbb{N},\ p_{j,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^p) \text{ for } \varepsilon \to 0 \right\}. \end{aligned}$$

With those definitions, we have:

Lemma 7 *$\mathcal{F}(\mathcal{D}(\Omega))$ is a subalgebra of $\mathcal{D}(\Omega)^{(0,1]}$ and $\mathcal{N}(\mathcal{D}(\Omega))$ an ideal of $\mathcal{F}(\mathcal{D}(\Omega))$.*

The factor space $\mathcal{G}_\mathcal{D}(\Omega) = \mathcal{F}(\mathcal{D}(\Omega))/\mathcal{N}(\mathcal{D}(\Omega))$ appears to be a natural space of generalized functions with compact supports. The algebra $\mathcal{G}_\mathcal{D}(\Omega)$ does not depend on the particular choice of the sequence $(K_j)_{j\in\mathbb{N}}$ Moreover, due to the properties of the family $(p_{j,l})$ we have:

Lemma 8 *The spaces $\mathcal{G}_\mathcal{D}(\Omega)$ and $\mathcal{G}_C(\Omega)$ are isomorphic.*

Proof. The fundamental property involved is the following: for all $j \in \mathbb{N}$ and all $(f_\varepsilon)_\varepsilon \in \mathcal{F}_j(\Omega)$ we have

$$\forall l \in \mathbb{N},\ \ \forall j' \leq j,\ \ \forall j'' \geq j, \quad p_{j',l}(f_\varepsilon) \leq p_{j,l}(f_\varepsilon) = p_{j'',l}(f_\varepsilon)\,. \tag{2}$$

The last equality is true since $\operatorname{supp} f \subset K_j \subset K_{j''}$, for all $j'' \geq j$.

Relation (2) implies that $\mathcal{F}(\mathcal{D}(\Omega)) \subset \mathcal{F}(\mathrm{C}^\infty(\nleq))$ and $\mathcal{N}(\mathcal{D}(\Omega)) \subset \mathcal{N}(\mathrm{C}^\infty(\nleq))$. Let us show the first inclusion. Consider $(f_\varepsilon)_\varepsilon$ in some $\mathcal{F}_j(\Omega)$. Then, for all $l \in \mathbb{N}$, there exists $q \in \mathbb{N}$ such that: $\forall j \in \mathbb{N}$, $p_{j,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{-q})$ for $\varepsilon \to 0$. It follows that $\forall K \Subset \Omega$, $p_{K,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{-q})$ since the sequence $(K_j)_{j\in\mathbb{N}}$ exhausts K.

Those two inclusions implies that the map

$$\iota:\ \mathcal{G}_{\mathcal{D}}(\Omega) \to \mathcal{G}(\Omega)\,, \qquad (f_\varepsilon)_\varepsilon + \mathcal{N}(\mathcal{D}(\Omega)) \mapsto (f_\varepsilon)_\varepsilon + \mathcal{N}(\mathrm{C}^\infty(\nleq))$$

is well defined with $\iota(\mathcal{G}_{\mathcal{D}}(\Omega)) \subset \mathcal{G}_C(\Omega)$.

It remains to show that the map ι is bijective. Indeed, if $(f_\varepsilon)_\varepsilon \in \mathcal{N}(\mathrm{C}^\infty(\nleq))$ with $(f_\varepsilon)_\varepsilon \in \mathcal{F}_j(\Omega)$, we have $(f_\varepsilon)_\varepsilon \in \mathcal{N}_j(\Omega)$ and $(f_\varepsilon)_\varepsilon \in \mathcal{N}(\mathcal{D}(\Omega))$. Injectivity follows. Conversely, take $g \in \mathcal{G}_C(\Omega)$. According to lemma 6, there exists a compact K and a representative $(g_\varepsilon)_\varepsilon$ of g such that $\operatorname{supp} g_\varepsilon \subset K$, for all ε. We observe that K is included in some K_j and then that $(g_\varepsilon)_\varepsilon \in \mathcal{F}_j(\Omega)$. finally, $\iota((g_\varepsilon)_\varepsilon + \mathcal{N}(\mathcal{D}(\Omega))) = g$. ■

2.3 Embeddings

The space $\mathrm{C}^\infty(\mathbb{R}^d)$ $(d \in \mathbb{N})$ is embedded in $\mathcal{G}(\mathbb{R}^d)$ by the canonical map

$$\sigma:\ \mathrm{C}^\infty\left(\mathbb{R}^d\right) \to \mathcal{G}\left(\mathbb{R}^d\right) \quad f \to (f_\varepsilon)_\varepsilon + \mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right), \quad \text{with } f_\varepsilon = f \text{ for all } \varepsilon \in (0,1]$$

which is an injective homomorphism of algebras.

Moreover, the construction of $\mathcal{G}(\mathbb{R}^d)$ permits to embed the space $\mathcal{D}'(\mathbb{R}^d)$ by means of convolution with suitable mollifiers. We follow in this paper the ideas of [15].

Lemma 9 *There exists a net of mollifiers $(\theta_\varepsilon)_\varepsilon \in \mathcal{D}(\mathbb{R}^d)^{(0,1]}$ for all ε, such that for all $k \in \mathbb{N}$*

$$\int \theta_\varepsilon(x)\,\mathrm{d}x = 1 + \mathrm{O}\left(\varepsilon^k\right) \textit{ for } \varepsilon \to 0, \tag{3}$$

$$\forall m \in \mathbb{N}^d \backslash \{0\}\,,\ \int x^m \theta_\varepsilon(x)\,\mathrm{d}x = \mathrm{O}\left(\varepsilon^k\right) \textit{ for } \varepsilon \to 0. \tag{4}$$

Such a net is built in the following way: Consider $\rho \in \mathcal{S}(\mathbb{R}^d)$ such that $\int \rho(x)\,\mathrm{d}x = 1$, $\int x^m \rho(x)\,\mathrm{d}x = 0$ for all $m \in \mathbb{N}^d \backslash \{0\}$ and $\kappa \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \leq \kappa \leq 1$, $\kappa = 1$ on $[-1,1]^d$ and $\kappa = 0$ on $\mathbb{R}^d \backslash [-2,2]^d$. Then $(\theta_\varepsilon)_\varepsilon$ defined by

$$\forall \varepsilon \in (0,1]\,,\ \ \forall x \in \mathbb{R}^d, \quad \theta_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right)\kappa(x\,|\ln \varepsilon|)$$

satisfies conditions of lemma 9.

Proposition 10 *With notations of lemma 9, the map*

$$\iota:\ \mathcal{D}'\left(\mathbb{R}^d\right) \to \mathcal{G}\left(\mathbb{R}^d\right) \quad T \mapsto (T * \theta_\varepsilon)_\varepsilon + \mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right)$$

is an injective homomorphism of vector spaces. Moreover $\iota_{|\mathrm{C}^\infty(\Omega)} = \sigma$.

This proposition asserts that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{C}^\infty\left(\mathbb{R}^d\right) & \longrightarrow & \mathcal{D}'\left(\mathbb{R}^d\right) \\ & \searrow\sigma & \downarrow\iota \\ & & \mathcal{G}\left(\mathbb{R}^d\right) \end{array}$$

2.4 Generalized Integral operators

We collect here results about generalized integral operators. We refer the reader to [1] and [6] for details.

Definition 11 *Let H be in $\mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$. The integral operator of Kernel H is the map $\widetilde{H}$ defined by*

$$\widetilde{H} : \mathcal{G}_C(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^m) : f \mapsto \widetilde{H}(f) \ \text{ with } \widetilde{H}(f) = \left[\left(x \mapsto \int H_\varepsilon(x,y) f(y)\,\mathrm{d}y\right)_\varepsilon\right]$$

where $(H_\varepsilon)_\varepsilon$ is any representative of H.

Note that in the above mentioned references, the generalized function H satisfies some additive condition such as being properly supported. This assumption is not needed in this paper, since we consider operators on $\mathcal{G}_C(\mathbb{R}^n)$: the integral which appears in definition 11 is performed on a compact set.

Proposition 12 *With the notations of definition 11 the operator $\widetilde{H}$ defines a linear mapping from $\mathcal{G}_C(\mathbb{R}^n)$ to $\mathcal{G}(\mathbb{R}^m)$ continuous for the respective sharp topologies of $\mathcal{G}_C(\mathbb{R}^n)$ and $\mathcal{G}(\mathbb{R}^m)$. Moreover the map*

$$\mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n) \to \mathcal{L}(\mathcal{G}_C(\mathbb{R}^n), \mathcal{G}(\mathbb{R}^m)) \qquad H \mapsto \widetilde{H}$$

is injective.

In other words, the map $\widetilde{H}$ is characterized by the kernel H

$$\widetilde{H} = 0 \text{ in } \mathcal{L}(\mathcal{G}_C(\mathbb{R}^n), \mathcal{G}(\mathbb{R}^m)) \Leftrightarrow H = 0 \text{ in } \mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n).$$

3 Spaces of generalized functions with slow growth

In the sequel, we need to consider some subspaces of $\mathcal{G}(\Omega)$ with restrictive conditions of growth with respect to $1/\varepsilon$ when the l index of the families of seminorms is involved, that is the index related to derivatives. We show that these spaces give a good framework for extension of linear maps and for convolution of generalized functions. These are essential properties for our result.

3.1 Definitions

Set

$$\mathcal{F}_{\mathcal{L}_0}(\mathrm{C}^\infty(\Omega)) = \Big\{ (f_\varepsilon)_\varepsilon \in \mathcal{F}(\Omega)^{(0,1]} \,\Big|\, \forall K \Subset \Omega,\ \exists q \in \mathbb{N}^{\mathbb{N}},\ \text{with } \lim_{l \to +\infty} (q(l)/l) = 0$$
$$\forall l \in \mathbb{N}, \quad p_{K,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-q(l)}\right) \text{ for } \varepsilon \to 0 \Big\}.$$

$$\mathcal{F}_{\mathcal{L}_1}(\mathrm{C}^\infty(\Omega)) = \Big\{ (f_\varepsilon)_\varepsilon \in \mathcal{F}(\Omega)^{(0,1]} \,\Big|\, \forall K \Subset \Omega,\ \exists q \in \mathbb{N}^{\mathbb{N}},\ \text{with } \limsup_{l \to +\infty} (q(l)/l) < 1$$
$$\forall l \in \mathbb{N}, \quad p_{K,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-q(l)}\right) \text{ for } \varepsilon \to 0 \Big\}.$$

Lemma 13
i. $\mathcal{F}_{\mathcal{L}_0}(\mathrm{C}^\infty(\Omega))$ *is a subalgebra of* $\mathcal{F}(\mathrm{C}^\infty(\Omega))$.
ii. $\mathcal{F}_{\mathcal{L}_1}(\mathrm{C}^\infty(\Omega))$ *is a submodulus of* $\mathcal{F}(\mathrm{C}^\infty(\Omega))$.

Proof. Take $(f_\varepsilon)_\varepsilon$ and $(g_\varepsilon)_\varepsilon$ in $\mathcal{F}_{\mathcal{L}_0}(\mathrm{C}^\infty(\Omega))$ (*resp.* $\mathcal{F}_{\mathcal{L}_1}(\mathrm{C}^\infty(\Omega))$), $K \Subset \Omega$, q_f and q_g the corresponding sequences with $\lim_{l\to+\infty}(q_h(l)/l) = 0$ (*resp.* $r_h = \limsup_{l\to+\infty}(q_h(l)/l) < 1$) for $h = f, g$. Define

$$q(\cdot) = \max(q_f(\cdot), q_g(\cdot)) \qquad r = \max(r_f, r_g) < 1, \text{ for the } \textit{resp.} \text{ case.}$$

For $h = f, g$ we have $p_{K,l}(h_\varepsilon) = \mathrm{O}\left(\varepsilon^{-q(l)}\right)$ for $\varepsilon \to 0$ and $p_{K,l}(f_\varepsilon + g_\varepsilon) = \mathrm{O}\left(\varepsilon^{-q(l)}\right)$ for $\varepsilon \to 0$.

For $(c_\varepsilon)_\varepsilon \in \mathcal{F}(\mathbb{R})$, there exists q_c such that $|c_\varepsilon| = \mathrm{O}(\varepsilon^{-q_c})$. Then $p_{K,l}(c_\varepsilon f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-(q_c+q(l))}\right)$ with $\lim_{l\to+\infty}((q_c + q(l))/l) = 0$ (*resp.* $\limsup_{l\to+\infty}((q_c + q(l))/l) < 1$). Thus $\mathcal{F}_{\mathcal{L}_0}(\mathrm{C}^\infty(\Omega))$ (*resp.* $\mathcal{F}_{\mathcal{L}_1}(\mathrm{C}^\infty(\Omega))$) are submodulus of $\mathcal{F}(\mathrm{C}^\infty(\Omega))$.

For $(f_\varepsilon)_\varepsilon$ and $(g_\varepsilon)_\varepsilon)$ in $\mathcal{F}_{\mathcal{L}_0}(\mathrm{C}^\infty(\Omega))$, there exists $C > 0$ such that

$$p_{K,l}(f_\varepsilon g_\varepsilon) \leq C p_{K,l}(f_\varepsilon)\, p_{K,l}(g_\varepsilon).$$

Consequently, $p_{K,l}(f_\varepsilon g_\varepsilon) = \mathrm{O}\left(\varepsilon^{-2q(l)}\right)$ for $\varepsilon \to 0$, with $\lim_{l\to+\infty}(2q(l)/l) = 0$. Thus $\mathcal{F}_{\mathcal{L}_0}(\mathrm{C}^\infty(\Omega))$ is a subalgebra of $\mathcal{F}(\mathrm{C}^\infty(\Omega))$. ■

Consequently, we can consider the following subalgebra (*resp.* submodulus)

$$\mathcal{G}_{\mathcal{L}_0}(\Omega) = \mathcal{F}_{\mathcal{L}_0}(\mathrm{C}^\infty(\Omega))/\mathcal{N}(\mathrm{C}^\infty(\Omega)) \quad (\textit{resp. } \mathcal{G}_{\mathcal{L}_1}(\Omega) = \mathcal{F}_{\mathcal{L}_1}(\mathrm{C}^\infty(\Omega))/\mathcal{N}(\mathrm{C}^\infty(\Omega)))$$

of $\mathcal{G}(\Omega)$.

Remark 14 *Some spaces with more restrictive conditions have already been considered (See e.g. [12], [16]). Set*

$$\mathcal{F}^\infty(\mathrm{C}^\infty(\Omega)) = \left\{(f_\varepsilon)_\varepsilon \in \mathcal{F}(\Omega)^{(0,1]} \,\middle|\, \forall K \Subset \Omega,\ \exists q \in \mathbb{N},\ \forall l \in \mathbb{N},\ p_{K,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-q}\right) \text{ for } \varepsilon \to 0\right\}.$$

$\mathcal{F}^\infty(\mathrm{C}^\infty(\Omega))$ *turns to be a subalgebra of* $\mathcal{F}_{\mathcal{L}_0}(\mathrm{C}^\infty(\Omega))$, $\mathcal{F}_{\mathcal{L}_1}(\mathrm{C}^\infty(\Omega))$ *and*

$$\mathcal{G}^\infty(\Omega) = \mathcal{F}^\infty(\mathrm{C}^\infty(\Omega))/\mathcal{N}(\mathrm{C}^\infty(\Omega))$$

a subalgebra of $\mathcal{G}_{\mathcal{L}}(\Omega)$, $\mathcal{G}_{\mathcal{L}_1}(\Omega)$ *and* $\mathcal{G}(\Omega)$. *For the local analysis or microlocal analysis of generalized functions, the* $\mathcal{G}^\infty$ *regularity plays the role of the* C^∞*' one for distributions [15] [13]. Our spaces* $\mathcal{G}_{\mathcal{L}_0}(\Omega)$ *and* $\mathcal{G}_{\mathcal{L}_1,C}(\Omega)$ *give new types of regularity for generalized functions. This will be studied in a forthcoming paper.*

Notation 15 *We shall note* $\mathcal{G}_C^\infty(\Omega)$ *(resp.* $\mathcal{G}_{\mathcal{L}_0,C}(\Omega)$, $\mathcal{G}_{\mathcal{L}_1,C}(\Omega)$*) the subspace of compactly supported elements of* $\mathcal{G}^\infty(\Omega)$ *(resp.* $\mathcal{G}_{\mathcal{L}_0}(\Omega)$, $\mathcal{G}_{\mathcal{L}_1}(\Omega)$*).*

3.2 Fundamental lemma

Lemma 16 *Let d be an integer and* $(\theta_\varepsilon)_\varepsilon \in \mathcal{D}\left(\mathbb{R}^d\right)^{(0,1]}$ *a net of mollifiers satisfying conditions (3) and (4). For any* $(g_\varepsilon)_\varepsilon \in \mathcal{F}_{\mathcal{L}_1}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right)$ *we have*

$$(g_\varepsilon * \theta_\varepsilon - g_\varepsilon)_\varepsilon \in \mathcal{N}\left(C^\infty\left(\mathbb{R}^d\right)\right). \tag{5}$$

Proof. We shall prove this lemma in the case $d = 1$, the general case only differs by more complicate algebraic expressions.

Fix $(g_\varepsilon)_\varepsilon \in F_{\mathcal{L}}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right)$, K a compact of $\mathbb{R}$ and set $\Delta_\varepsilon = g_\varepsilon * \theta_\varepsilon - g_\varepsilon$ for $\varepsilon \in (0,1]$. Writing $\int \theta_\varepsilon(x)\,\mathrm{d}x = 1 + \mathcal{N}_\varepsilon$ with $(\mathcal{N}_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$ we get

$$\Delta_\varepsilon(y) = \int g_\varepsilon(y - x)\theta_\varepsilon(x)\,\mathrm{d}x - g_\varepsilon(y) = \int (g_\varepsilon(y - x) - g_\varepsilon(y))\,\theta_\varepsilon(x)\,\mathrm{d}x + \mathcal{N}_\varepsilon g_\varepsilon(y).$$

The integration is performed on the compact set $\operatorname{supp}\theta_\varepsilon = [-2/|\ln\varepsilon|, 2/|\ln\varepsilon|]$.

Let m be an integer. For each $i\in\mathbb{N}$,there exists an integer $q(i)$ such that

$$\sup_{\xi\in K'}\left|g_\varepsilon^{(i)}(\xi)\right| = \mathrm{O}\left(\varepsilon^{-q(i)}\right) \quad \text{for } \varepsilon\to 0$$

with $\limsup_{i\to+\infty}(q(i)/i) < 1$ and K' is a compact such that $[y-1,y+1]\subset K'$ for all $y\in K$.

As $\limsup_{i\to+\infty}(q(i)/i) < 1$, we get $\lim_{i\to+\infty}(i-l(i)) = +\infty$, and there exists an integer k such that $k-l(k) > m$. Taylor's formula gives

$$g_\varepsilon(y-x) - g_\varepsilon(y) = \sum_{i=1}^{k-1}\frac{(-x)^i}{i!}g_\varepsilon^{(i)}(y) + \frac{(-x)^{k-1}}{(k-1)!}\int_0^1 g_\varepsilon^{(k)}(y-ux)(1-u)^{k-1}\,\mathrm{d}u$$

and

$$\Delta_\varepsilon(y) = \underbrace{\sum_{i=1}^{k-1}\frac{(-1)^i}{i!}g_\varepsilon^{(i)}(y)\int x^i\theta_\varepsilon(x)\,\mathrm{d}x}_{P_\varepsilon(k,y)}$$
$$+\underbrace{\int_{-2/|\ln\varepsilon|}^{2/|\ln\varepsilon|}\frac{(-x)^{k-1}}{(k-1)!}\int_0^1 g_\varepsilon^{(k)}(y-ux)(1-u)^{k-1}\,\mathrm{d}u\,\theta_\varepsilon(x)\,\mathrm{d}x}_{R_\varepsilon(k,y)} + \mathcal{N}_\varepsilon g_\varepsilon^{(k)}(y).$$

According to lemma 9, we have $\left(\int x^i\theta_\varepsilon(x)\,\mathrm{d}x\right)_\varepsilon \in\mathcal{N}(\mathbb{R})$ and consequently

$$\forall i\in\{0,\dots,k-1\},\quad \int x^i\theta_\varepsilon(x)\,\mathrm{d}x = \mathrm{O}\left(\varepsilon^{m+q(i)}\right) \quad \text{for } \varepsilon\to 0.$$

We get

$$P_\varepsilon(k,y) = \mathrm{O}(\varepsilon^m) \quad \text{for } \varepsilon\to 0.$$

Using the definition of θ_ε, we have

$$R_\varepsilon(k,y) = \frac{1}{\varepsilon}\int_{-2/|\ln\varepsilon|}^{2/|\ln\varepsilon|}\frac{(-x)^{k-1}}{(k-1)!}\left(\int_0^1 g_\varepsilon^{(k)}(y-ux)(1-u)^{k-1}\,\mathrm{d}u\right)\rho\left(\frac{x}{\varepsilon}\right)\chi(x|\ln\varepsilon|)\,\mathrm{d}x.$$

Setting $v = x/\varepsilon$ we get

$$R_\varepsilon(k,y) = \frac{\varepsilon^{k-1}}{(k-1)!}\int_{-2/(\varepsilon|\ln\varepsilon|)}^{2/(\varepsilon|\ln\varepsilon|)}(-v)^{k-1}\left(\int_0^1 g_\varepsilon^{(k)}(y-\varepsilon uv)(1-u)^{k-1}\,\mathrm{d}u\right)\rho(v)\,\chi(\varepsilon|\ln\varepsilon|v)\,\mathrm{d}v.$$

For $(u,v)\in[0,1]\times[-2/(\varepsilon|\ln\varepsilon|), 2/(\varepsilon|\ln\varepsilon|)]$, we have $y-\varepsilon uv\in[y-1,y+1]$ for ε small enough. Then, for $y\in K$, $y-\varepsilon uv$ lies in a compact K' for (u,v) in the domain of integration.

It follows

$$\begin{aligned}
|R_\varepsilon(k,y)| &\le \frac{\varepsilon^{k-1}}{(k-1)!}\sup_{\xi\in K'}\left|g_\varepsilon^{(k)}(\xi)\right|\int_{-2/(\varepsilon|\ln\varepsilon|)}^{2/(\varepsilon|\ln\varepsilon|)}|v|^{k-1}|\rho(v)|\,\mathrm{d}v,\\
&\le \frac{\varepsilon^{k-1}}{(k-1)!}\sup_{\xi\in K'}\left|g_\varepsilon^{(k)}(\xi)\right|\int_{-\infty}^{+\infty}|v|^{k-1}|\rho(v)|\,\mathrm{d}v,\\
&\le C\sup_{\xi\in K'}\left|g_\varepsilon^{(k)}(\xi)\right|\varepsilon^{k-1}\quad (C>0)
\end{aligned}$$

The constant C depends only on the integer k and ρ. By assumption on k, we get

$$\sup_{y\in K}|R_\varepsilon(k,y)| = \mathrm{O}\left(\varepsilon^m\right) \text{ for } \varepsilon \to 0.$$

Summering all results, we get $\sup_{y\in K}\Delta_\varepsilon(y) = \mathrm{O}\left(\varepsilon^m\right)$ for $\varepsilon \to 0$.

As $(\Delta_\varepsilon)_\varepsilon \in \mathcal{F}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right)$ and $\sup_{y\in K}\Delta_\varepsilon(y) = \mathrm{O}\left(\varepsilon^m\right)$ for $\varepsilon \to 0$, for all $m > 0$ and $K \Subset \mathbb{R}$, we can conclude that $(\Delta_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^d\right)\right)$ without estimating the derivatives by using theorem 1.2.3. of [7]. ■

Remark 17 *Let us fix a net of mollifiers $(\theta_\varepsilon)_\varepsilon$ satisfying conditions (3) and (4) to embed $\mathcal{D}'\left(\mathbb{R}^d\right)$ in $\mathcal{G}\left(\mathbb{R}^d\right)$. Relation (5) shows that $[(\theta_\varepsilon)_\varepsilon]$ is plays the role of identity for convolution in $\mathcal{G}_{\mathcal{L}_0}\left(\mathbb{R}^d\right)$ and $\mathcal{G}_{\mathcal{L}_1}\left(\mathbb{R}^d\right)$, whereas this is not true for $\mathcal{G}\left(\mathbb{R}^d\right)$. This is an essential feature of these new spaces. (See also example 22 below.)*

4 Schwartz type theorem

4.1 Extension of linear maps

Nets of maps $(L_\varepsilon)_\varepsilon$ between two topological algebras having some good growth properties with respect to the parameter ε can be canonically extended to the respective Colombeau spaces based on algebras as it is shown in [5], [4], [7] for examples. We are going to introduce here some new notions.

We uses the notations of 2.2, specially

$$\mathcal{D}_J\left(\mathbb{R}^n\right) = \left\{f \in \mathcal{D}\left(\mathbb{R}^n\right) \mid \operatorname{supp} f \subset K_J\right\},$$

where $(K_J)_{J\in\mathbb{N}}$ is a sequence of compacts exhausting $\mathbb{R}^n$, and $\mathcal{D}_J\left(\mathbb{R}^n\right)$ is endowed with the family of semi norms $p_{J,l}\left(f\right) = \sup_{|\alpha|\leq l,\, x\in K_J}\left|\partial^\alpha f\left(x\right)\right|$.

Definition 18 *Let J be an integer and $(L_\varepsilon)_\varepsilon \in \mathcal{L}\left(\mathcal{D}_J\left(\mathbb{R}^n\right), \mathrm{C}^\infty\left(\mathbb{R}^m\right)\right)^{(0,1]}$ be a net of linear maps.*
i. We say that $(L_\varepsilon)_\varepsilon$ is moderate *if*

$$\forall K \Subset \mathbb{R}^m,\ \forall l \in \mathbb{N},\ \exists\left(C_\varepsilon\right)_\varepsilon \in \mathcal{F}\left(\mathbb{R}_+\right),\ \exists l' \in \mathbb{N},$$
$$\forall f \in \mathcal{D}_J\left(\mathbb{R}^n\right) \quad p_{K,l}\left(L_\varepsilon\left(f\right)\right) \leq C_\varepsilon p_{J,l'}\left(f\right) \ \textit{(for } \varepsilon \textit{ small enough).}$$

ii. We say that $(L_\varepsilon)_\varepsilon$ is strongly moderate *if*

$$\forall K \Subset \mathbb{R}^m,\ \exists\lambda \in \mathbb{N}^\mathbb{N} \textit{ with } \lambda(l) = \mathrm{O}(l) \textit{ for } l \to +\infty,\ \exists r \in \mathbb{N}^\mathbb{N} \textit{ with } \limsup_{l\to+\infty}\left(r(l)/l\right) < 1,$$
$$\forall l \in \mathbb{N},\ \exists C \in \mathbb{R}_+,\ \forall f \in \mathcal{D}_J\left(\mathbb{R}^n\right),\ \ p_{K,l}\left(L_\varepsilon\left(f\right)\right) \leq C\varepsilon^{-r(l)}p_{J,\lambda(l)}\left(f\right) \ \textit{(for } \varepsilon \textit{ small enough).}$$

In the strong moderation, the growth of $p_{K,l}\left(L_\varepsilon\left(f\right)\right)$ with respect to the index l is controlled by the sequence $\lambda\left(\cdot\right)$ which grows at most like l. and by the sequence $r(l)$.

As our main result is based on linear maps from $\mathcal{D}\left(\mathbb{R}^n\right)$ to $\mathrm{C}^\infty\left(\mathbb{R}^m\right)$ we need one further extension:

Definition 19 *A net of maps $(L_\varepsilon)_\varepsilon \in \left(\mathcal{L}\left(\mathcal{D}\left(\mathbb{R}^n\right), \mathrm{C}^\infty\left(\mathbb{R}^m\right)\right)^{(0,1]}\right)$ is* moderate *(resp.* strongly moderate*) if for every $J \in \mathbb{N}$, the restriction $\left(L_{\varepsilon|\mathcal{D}_J(\mathbb{R}^n)}\right) \in \mathcal{L}\left(\mathcal{D}_J\left(\mathbb{R}^n\right), \mathrm{C}^\infty\left(\mathbb{R}^m\right)\right)^{(0,1]}$ is moderate (resp. strongly moderate) in the sense of definition 18.*

Proposition 20 *Any moderate net* $(L_\varepsilon) \in \left(\mathcal{L}\left(\mathcal{D}\left(\mathbb{R}^n\right), \mathrm{C}^\infty\left(\mathbb{R}^m\right)\right)^{(0,1]}\right)$*, in the sense of definition 19, admits a canonical extension* $L \in \mathcal{L}\left(\mathcal{G}_C\left(\mathbb{R}^n\right), \mathcal{G}\left(\mathbb{R}^m\right)\right)$ *defined by*

$$L\left([(f_\varepsilon)]\right) = L_\varepsilon\left(f_\varepsilon\right) + \mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^m\right)\right). \tag{6}$$

Moreover, if the net (L_ε) *is strongly moderate,* $L\left(\mathcal{G}_{\mathcal{L}0,C}\left(\mathbb{R}^n\right)\right)$ *is included in* $\mathcal{G}_{\mathcal{L}_1}\left(\mathbb{R}^m\right)$

Proof. Fix $K \Subset \mathbb{R}^m$, $l \in \mathbb{N}$ and let $(f_\varepsilon)_\varepsilon$ be in $\mathcal{F}_\mathcal{D}\left(\mathbb{R}^n\right)$. There exists $J \in \mathbb{N}$ such that $(f_\varepsilon)_\varepsilon \in \mathcal{F}_J\left(\mathbb{R}^n\right)$ and according to the definition of moderate nets, we get $(C_\varepsilon)_\varepsilon \in \mathcal{F}\left(\mathbb{R}_+\right)$ and $l' \in \mathbb{N}$ such that

$$p_{K,l}\left(L_\varepsilon\left(f_\varepsilon\right)\right) \leq C_\varepsilon p_{J,l'}\left(f_\varepsilon\right), \text{ for } \varepsilon \text{ small enough.} \tag{7}$$

Inequality (7) leads to $\left(L_\varepsilon\left(f_\varepsilon\right)\right)_\varepsilon \in \mathcal{F}\left(\mathrm{C}^\infty\left(\mathbb{R}^m\right)\right)$. Moreover, if $(f_\varepsilon)_\varepsilon$ belongs to $\mathcal{N}_\mathcal{D}\left(\mathbb{R}^n\right)$ the same inequality implies that $\left(L_\varepsilon\left(f_\varepsilon\right)\right)_\varepsilon \in \mathcal{N}\left(\mathrm{C}^\infty\left(\mathbb{R}^m\right)\right)$. Those two properties shows that L is well defined by formula (6).

Now, suppose that $(L_\varepsilon)_\varepsilon$ is strongly moderate and consider $(f_\varepsilon)_\varepsilon \in \mathcal{F}_{\mathcal{L}_0}\left(\mathrm{C}^\infty\left(\mathbb{R}^n\right)\right) \cap \mathcal{F}_J\left(\mathbb{R}^n\right)$. Fix $K \Subset \mathbb{R}^m$. There exists a sequence $\lambda \in \mathbb{N}^\mathbb{N}$, with $\lambda(l) = \mathrm{O}(l)$ for $l \to +\infty$, and a sequence $r \in \mathbb{N}^\mathbb{N}$ with $\limsup_{l\to+\infty}\left(r(l)/l\right) < 1$ such that

$$\forall l \in \mathbb{N},\ \ \exists C \in \mathbb{R}_+,\quad p_{K,l}\left(L_\varepsilon\left(f_\varepsilon\right)\right) \leq C\varepsilon^{-r(l)} p_{J,\lambda(l)}\left(f_\varepsilon\right)\ \text{ (for } \varepsilon \text{ small enough).}$$

As $(f_\varepsilon)_\varepsilon$ is in $\mathcal{F}_{\mathcal{L}_0}\left(\mathrm{C}^\infty\left(\mathbb{R}^n\right)\right)$, there exists a sequence $q \in \mathbb{N}^\mathbb{N}$, with $\lim_{\lambda\to+\infty}\left(q(\lambda)/\lambda\right) = 0$ such that

$$\forall \lambda \in \mathbb{N},\qquad p_{J,\lambda}\left(f_\varepsilon\right) = \mathrm{O}\left(\varepsilon^{-q(\lambda)}\right)\ \text{ for } \varepsilon \to 0.$$

We get that

$$\forall l \in \mathbb{N},\qquad p_{K,l}\left(L_\varepsilon\left(f_\varepsilon\right)\right) = \mathrm{O}\left(\varepsilon^{-q_1(l)}\right)\ \text{ for } \varepsilon \to 0,\ \text{ with } q_1\left(l\right) = r(l) + q\left(\lambda(l)\right).$$

If $\lambda(l)$ is bounded, we get immediately that $q_1\left(l\right)/l = \mathrm{o}\left(1\right)$ for $l \to +\infty$. If $\lambda(l)$ is not bounded, we write for l such that $\lambda(l) \neq 0$.

$$\frac{q_1\left(l\right)}{l} = \frac{r(l)}{l} + \frac{q\left(\lambda(l)\right)}{\lambda(l)}\frac{\lambda(l)}{l}$$

Since $\lambda(l)/l$ is bounded and $q\left(l\right)/l = \mathrm{o}\left(1\right)$, we get that $\frac{q(\lambda(l))}{\lambda(l)}\frac{\lambda(l)}{l} = \mathrm{o}\left(1\right)$ for $l \to +\infty$. This gives that $\limsup_{l\to+\infty}\left(q_1(l)/l\right) < 1$ and $\left(L_\varepsilon\left(f_\varepsilon\right)\right)_\varepsilon \in \mathcal{F}_{\mathcal{L}_1}\left(\mathrm{C}^\infty\left(\Omega\right)\right)$ and shows last assertion. ■

4.2 Main theorem

Theorem 21 *Let* $(L_\varepsilon)_\varepsilon \in \mathcal{L}\left(\mathcal{D}(\mathbb{R}^n), \mathrm{C}^\infty(\mathbb{R}^m)\right)^{(0,1]}$ *be a net of strongly moderate continuous linear maps and* $L \in \mathcal{L}\left(\mathcal{G}_C(\mathbb{R}^n), \mathcal{G}(\mathbb{R}^m)\right)^{(0,1]}$ *its canonical extension. There exists* $H_L \in \mathcal{G}\left(\mathbb{R}^m \times \mathbb{R}^n\right)$ *such that*

$$\forall f \in \mathcal{G}_{\mathcal{L}_0,C}\left(\mathbb{R}^n\right)),\quad L\left(f\right)(x) = \int H_L\left(x,y\right) f(y)\,\mathrm{d}y.$$

In other words, L restricted to $\mathcal{G}_{\mathcal{L}0,C}\left(\Omega\right))$ can be represented by a kernel H_L. The fact that the equality is only valid in $\mathcal{G}_{\mathcal{L}0,C}\left(\mathbb{R}^n\right))$ is not surprising. The structure of the theorem is similar as Schwartz'one: f belongs to a "smaller" type of space as H_L and $L\left(f\right)$, which both belongs to the same kind of space.

Example 22 *Remark 17 and relation (5) shows also that the identity map of $\mathcal{G}_{\mathcal{L}_0,C}(\mathbb{R}^n))$ admits as kernel*

$$\Phi = cl\left(\left((x,y) \mapsto \varphi_\varepsilon(x-y)\right)_\varepsilon\right) \tag{8}$$

where $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$ is any net of mollifiers satisfying conditions (3) and (4) of lemma 9.

This example shows also that, in general, we don't have uniqueness in theorem 21, but a so called *weak uniqueness*. In our example, any net $(\varphi_\varepsilon)_\varepsilon$ of mollifiers satisfies $\varphi_\varepsilon \to \delta$ in $\mathcal{D}'$ for $\varepsilon \to 0$: Thus, kernels of the form (8) are associated in $\mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$ or weakly equal i.e. the difference of their representative tends to 0 in $\mathcal{D}'$ for $\varepsilon \to 0$. (See [4], [7], [11], [15] for further analysis of different associations in Colombeau type spaces.)

4.3 Link with the classical Schwartz theorem: Equality in generalized distribution sense

Let $\Lambda \in \mathcal{L}(\mathcal{D}(\mathbb{R}^n), \mathcal{D}'(\mathbb{R}^m))$ be continuous for the strong topology and consider the family of linear mappings $(L_\varepsilon)_{\varepsilon\in}$ defined by

$$L_\varepsilon : \mathcal{D}(\mathbb{R}^n) \to \mathrm{C}^\infty(\mathbb{R}^m) \qquad f \mapsto \Lambda(f) * \varphi_{\sqrt{\varepsilon}},$$

where $(\varphi_\varepsilon)_\varepsilon$ is a family of mollifiers satisfying conditions (3) and (4) of lemma 9. We have:

Proposition 23
i. For all $\varepsilon \in (0,1]$, L_ε is continuous for the usual topologies of $\mathcal{D}(\mathbb{R}^n)$ and $\mathrm{C}^\infty(\mathbb{R}^m)$.
ii. The net $(L_\varepsilon)_\varepsilon$ is strongly moderate.

Consequently, theorem 21 shows that the canonical extension L of the net $(L_\varepsilon)_\varepsilon$ admits a kernel H_L.

Proposition 24 *For all $f \in \mathcal{D}(\mathbb{R}^n)$, $\Lambda(f)$ is equal to $\widetilde{H}_L(f)$ in the* generalized distribution sense*, that is*

$$\forall \Phi \in \mathcal{D}(\mathbb{R}^m), \quad \langle \Lambda(f), \Phi \rangle = \left\langle \widetilde{H}_L(f), \Phi \right\rangle \text{ in } \overline{\mathbb{C}}.$$

This generalized distribution equality, introduced in [15], means in other words that, for all $k \in \mathbb{N}$,

$$\forall \Phi \in \mathcal{D}(\mathbb{R}^m), \ \langle \Lambda(f), \Phi \rangle - \int \left(\int H_{L,\varepsilon}(x,y) f(y) \, \mathrm{d}y \right) \Phi(x) \, \mathrm{d}x = \mathrm{O}\left(\varepsilon^k\right), \text{ for } \varepsilon \to 0, \tag{9}$$

where $(H_{L,\varepsilon})_\varepsilon$ is any representative of H_L.

In particular, this result implies that $\Lambda(f)$ and $\widetilde{H}_L(f)$ are associated or weakly equal, *id est*

$$\left\{ x \mapsto \int H_{L,\varepsilon}(x,y) f(y) \, \mathrm{d}y \right\} \longrightarrow \Lambda(f) \text{ in } \mathcal{D}' \text{ for } \varepsilon \to 0.$$

5 Proofs of theorem 21 and propositions 23 and 24

5.1 Proof of theorem 21

Let us fix $(\varphi_\varepsilon)_\varepsilon \in (\mathcal{D}(\mathbb{R}^m))^{(0,1]}$ (*resp.* $(\psi_\varepsilon)_\varepsilon \in (\mathcal{D}(\mathbb{R}^m))^{(0,1]}$) a net of mollifiers satisfying conditions 3 and 4 of lemma 9. For all $y \in \mathbb{R}^n$ we define

$$\psi_{\varepsilon,\cdot} : \mathbb{R}^n \to \mathcal{D}(\mathbb{R}^n) \qquad y \mapsto \psi_{\varepsilon,y} = \{v \mapsto \psi_\varepsilon(y-v)\}.$$

For all $y \in \mathbb{R}^n$ and $\varepsilon \in (0,1]$, we set $\Psi_{\varepsilon,y} = L_\varepsilon(\psi_{\varepsilon,y})$.

Lemma 25 *The map*

$$\Psi_\varepsilon : \mathbb{R}^n \to \mathrm{C}^\infty(\mathbb{R}^m) \quad y \mapsto \Psi_{\varepsilon,y} = L_\varepsilon\left(\psi_{\varepsilon,y}\right)$$

is of class C^∞ *for all* $\varepsilon \in (0,1]$.

Proof. The map $(y,v) \mapsto \psi_\varepsilon(y-v)$ from $\mathbb{R}^{2n}$ to $\mathbb{R}$ is clearly of class C^∞. It follows that the map $\psi_{\varepsilon,\cdot} : y \mapsto \psi_{\varepsilon,y}$, considered as a map from $\mathbb{R}^n$ to $\mathrm{C}^\infty(\mathbb{R}^n)$, is C^∞ (see for example theorem 2.2.2 of [7]). As each $\psi_{\varepsilon,y}$ is compactly supported we can show that $\psi_{\varepsilon,\cdot}$ belongs in fact to $\mathrm{C}^\infty\left(\mathbb{R}^n, \mathcal{D}\left(\mathbb{R}^n\right)\right)$ by using local arguments. Since L_ε is linear and continuous it follows that Ψ_ε is C^∞. ■

Let us define, for all $\varepsilon \in (0,1]$ and $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$:

$$H_\varepsilon(x,y) = \left(\Psi_{\varepsilon,y} * \varphi_\varepsilon\right)(x) = \int L_\varepsilon\left(\psi_{\varepsilon,y}\right)(x-\lambda)\,\varphi_\varepsilon(\lambda)\,\mathrm{d}\lambda.$$

Note that, for all $\varepsilon \in (0,1]$, this integral is performed on the compact set $\operatorname{supp}\varphi_\varepsilon$.

Lemma 26 *For all* $\varepsilon \in (0,1]$, H_ε *is of class* C^∞ *and* $\left(H_\varepsilon\right)_\varepsilon \in \mathcal{F}\left(\mathbb{R}^m \times \mathbb{R}^n\right)$.

Proof. First, the map $g \mapsto g * \varphi_\varepsilon$ from $\mathrm{C}^\infty(\mathbb{R}^m)$ into itself is linear continuous and therefore C^∞. Using lemma 25, we get that the map $y \mapsto \left(\Psi_{\varepsilon,y} * \varphi_\varepsilon\right) = H_\varepsilon(\cdot,y)$ from $\mathbb{R}^n$ to $\mathrm{C}^\infty(\mathbb{R}^m)$ is C^∞. Using again theorem 2.2.2 of [7], we get that H_ε belongs to $\mathrm{C}^\infty(\mathbb{R}^{2n})$.

Consider K and K' two compact subsets of $\mathbb{R}^n$. Let us recall that the support of ψ_ε is compact and decreasing to $\{0\}$ when ε tends to 0. Then, there exists a compact set $K_\psi \subset \mathbb{R}^m$ such that, for all $\varepsilon \in (0,1]$, $\operatorname{supp}\psi_\varepsilon \subset K_\psi$ and $\operatorname{supp}\psi_{\varepsilon,y} \subset y - K\psi$. Moreover, we can find a compact K_J (notation are those of 4.1) such that

$$\forall \varepsilon \in (0,1], \quad \forall y \in K', \quad \psi_{\varepsilon,y} \in \mathcal{D}_J\left(\mathbb{R}^n\right).$$

and $p_{J,l}(\psi_{\varepsilon,y}) = p_{K_\psi,l}(\psi_\varepsilon)$, for all $\varepsilon \in (0,1]$.

Let now consider $(\alpha,\beta) \in \left(\mathbb{N}^n\right)^2$ and ∂^α (*resp.* ∂^β) the α-partial derivative (*resp.* β-partial derivative) with respect to the variable x (*resp.* y). Noticing that there exists a compact set $K_\varphi \subset \mathbb{R}^m$ such that, for all $\varepsilon \in (0,1]$, $\operatorname{supp}\varphi_{\varepsilon,y} \subset K_\varphi$ we get the existence of a constant C such that, for all $\varepsilon \in (0,1]$,

$$\begin{aligned}\forall (x,y) \in K \times K', \ \left|H_\varepsilon(x,y)\right| &\leq C \sup_{\xi \in K - K\varphi} \left|\partial^\beta L_\varepsilon\left(\psi_{\varepsilon,y}\right)(\xi)\right| \sup_{\xi \in K\varphi} \left|\partial^\alpha \varphi_\varepsilon(\xi)\right|, \\ &\leq C p_{K-K\varphi,|\beta|}\left(L_\varepsilon\left(\psi_{\varepsilon,y}\right)\right) p_{K\varphi,|\alpha|}\left(\varphi_\varepsilon\right).\end{aligned}$$

The moderateness of $\left(L_\varepsilon\right)_\varepsilon$ implies the existence of $l \in \mathbb{N}$ and $\left(C'_\varepsilon\right)_\varepsilon \in \mathcal{F}\left(\mathbb{R}_+\right)$ such that, for all $\varepsilon \in (0,1]$,

$$\forall (x,y) \in K \times K', \ \left|H_\varepsilon(x,y)\right| \leq C'_\varepsilon p_{J,l}\left(\psi_{\varepsilon,y}\right) p_{K\varphi,|\alpha|}\left(\varphi_\varepsilon\right) \leq C'_\varepsilon p_{K_\psi,l}(\psi_\varepsilon) p_{K\varphi,|\alpha|}\left(\varphi_\varepsilon\right).$$

The last inequality shows that $\left(p_{K \times K'|\alpha|+|\beta|}(H_\varepsilon)\right)_\varepsilon$ belongs to $\mathcal{F}\left(\mathbb{R}_+\right)$, this ending the proof. ■

For all $\left(f_\varepsilon\right)_\varepsilon$ in $\mathcal{F}\left(\mathcal{D}\left(\mathbb{R}^n\right)\right)$ (this set is defined by relation (1)) we can consider

$$\widetilde{H}_\varepsilon\left(f_\varepsilon\right)(x) = \int H_\varepsilon(x,y)\, f_\varepsilon(y)\,\mathrm{d}y = \int \left(\int L_\varepsilon\left(\psi_{\varepsilon,y}\right)(x-\lambda)\,\varphi_\varepsilon(\lambda)\,\mathrm{d}\lambda\right) f_\varepsilon(y)\,\mathrm{d}y.$$

since for all $\varepsilon \in (0,1]$, f_ε is compactly supported.

Lemma 27 *For all* $(f_\varepsilon)_\varepsilon$ *in* $\mathcal{F}(\mathcal{D}(\mathbb{R}^n))$*, we have*

$$\widetilde{H}_\varepsilon(f_\varepsilon)(x) = (L_\varepsilon(\psi_\varepsilon * f_\varepsilon) * \varphi_\varepsilon)(x).$$

Proof. Let $(f_\varepsilon)_\varepsilon$ be in $\mathcal{F}(\mathcal{D}(\mathbb{R}^n))$. For all $\varepsilon \in (0,1]$ and $x \in \mathbb{R}^m$, we have

$$\begin{aligned}
\widetilde{H}_\varepsilon(f_\varepsilon)(x) &= \int_{\operatorname{supp} f}\left(\int_{\operatorname{supp}\varphi_\varepsilon} L_\varepsilon(\psi_{\varepsilon,y})(x-\lambda)\,\varphi_\varepsilon(\lambda)\,\mathrm{d}\lambda\right) f_\varepsilon(y)\,\mathrm{d}y,\\
&= \int_{\operatorname{supp}\varphi_\varepsilon}\int_{\operatorname{supp} f} L_\varepsilon(\psi_{\varepsilon,y})(x-\lambda)\,\varphi_\varepsilon(\lambda)\,f_\varepsilon(y)\,\mathrm{d}\lambda\mathrm{d}y,\\
&= \int\left(\int L_\varepsilon(\psi_{\varepsilon,y})(x-\lambda)\,f_\varepsilon(y)\,\mathrm{d}y\right)\varphi_\varepsilon(\lambda)\,\mathrm{d}\lambda,
\end{aligned}$$

the two last equalities being true by Fubini's theorem, each integral being calculated on a compact set.

For all $\varepsilon \in (0,1]$ and $\xi \in \mathbb{R}^m$, we have the following equality:

$$\begin{aligned}
\int L_\varepsilon(\psi_{\varepsilon,y})(\xi)\,f_\varepsilon(y)\,\mathrm{d}y &= L_\varepsilon\left(v \mapsto \int \psi_{\varepsilon,y}(v)\,f_\varepsilon(y)\mathrm{d}y\right)(\xi),\\
&= L_\varepsilon\left(v \mapsto \int \psi_\varepsilon(y-v)\,f_\varepsilon(y)\mathrm{d}y\right)(\xi).
\end{aligned}$$

Indeed, the integrals under consideration in the above equalities are integrals of continuous functions on compact sets and can be considered as limits of Riemann sums in the spirit of [9] (Lemma 4.1.3, p. 89):

$$\begin{aligned}
\forall \xi \in \mathbb{R}^m,\quad \int L_\varepsilon(\psi_{\varepsilon,y})(\xi)\,f_\varepsilon(y)\,\mathrm{d}y &= \lim_{h\to 0}\sum_{k\in\mathbb{Z}} h^n L_\varepsilon(\psi_\varepsilon(kh-v))(\xi)\,f_\varepsilon(kh),\\
\forall v \in \mathbb{R}^n,\quad \int \psi_\varepsilon(y-v)\,f_\varepsilon(y)\mathrm{d}y &= \lim_{h\to 0}\sum_{k\in\mathbb{Z}} h^n \psi_\varepsilon(kh-v)\,f_\varepsilon kh).
\end{aligned}$$

As the mapping L_ε is linear, we have

$$L_\varepsilon\left(\sum_{k\in\mathbb{Z}}\psi_\varepsilon(kh-v)\,f_\varepsilon(kh)\right) = \sum_{k\in\mathbb{Z}} f_\varepsilon(kh)L_\varepsilon(\psi_\varepsilon(kh-v)),$$

as each $f_\varepsilon(kh)$ is a scalar: The function $\psi_{\varepsilon,y}$ is on the v variable, belonging to $\mathbb{R}^n$. By continuity of L_ε, we get

$$\begin{aligned}
L_\varepsilon\left(\int \psi_\varepsilon(y-v)\,f_\varepsilon(y)\mathrm{d}y\right)(\xi) &= L_\varepsilon\left(\lim_{h\to 0}\sum_{k\in\mathbb{Z}} h^n\psi_\varepsilon(kh-v)\,f_\varepsilon(kh)\right)(\xi),\\
&= \lim_{h\to 0}\left(\sum_{k\in\mathbb{Z}} f_\varepsilon(kh)L_\varepsilon(\psi_\varepsilon(kh-v))(\xi)\right),\\
&= \int L_\varepsilon(\psi_{\varepsilon,y})(\xi)\,f_\varepsilon(y)\,\mathrm{d}y.
\end{aligned}$$

Finally, we get for all $\varepsilon \in (0,1]$ and $\xi \in \mathbb{R}^m$,

$$\begin{aligned}
\int L_\varepsilon(\psi_{\varepsilon,y})(\xi)\,f_\varepsilon(y)\,\mathrm{d}y &= L_\varepsilon\left(\int \psi_\varepsilon(y-v)\,f_\varepsilon(y)\mathrm{d}y\right)(\xi),\\
&= L_\varepsilon(\psi_\varepsilon * f_\varepsilon)(\xi),
\end{aligned}$$

and

$$\widetilde{H}_{\varepsilon}\left(f_{\varepsilon}\right)(x)=\int L_{\varepsilon}\left(\psi_{\varepsilon}*f_{\varepsilon}\right)(x-\lambda)\,\varphi_{\varepsilon}(\lambda)\,\mathrm{d}\lambda=\left(L_{\varepsilon}\left(\psi_{\varepsilon}*f_{\varepsilon}\right)*\varphi_{\varepsilon}\right)(x). \tag{10}$$

■

We are now complete the proof of theorem 21. Set

$$H_L=\left(H_{\varepsilon}\right)_{\varepsilon}+\mathcal{N}\left(C^{\infty}\left(\mathbb{R}^{m+n}\right)\right)=\left((x,y)\mapsto\left(\Psi_{\varepsilon,y}*\varphi_{\varepsilon}\right)(x)\right)_{\varepsilon}+\mathcal{N}\left(C^{\infty}\left(\mathbb{R}^{m+n}\right)\right).$$

For all $\left(f_{\varepsilon}\right)_{\varepsilon}$ in $\mathcal{F}_{\mathcal{L}_0}\left(\mathcal{D}\left(\mathbb{R}^n\right)\right)$ we have

$$\widetilde{H}_L\left(\left[\left(f_{\varepsilon}\right)_{\varepsilon}\right]\right)=\left[\left(\widetilde{H}_{\varepsilon}\left(f_{\varepsilon}\right)\right)_{\varepsilon}\right]$$

by definition of the integral in $\mathcal{G}\left(\mathbb{R}^n\right)$. We have to compare $\left(\widetilde{H}_{\varepsilon}\left(f_{\varepsilon}\right)\right)_{\varepsilon}$ and $\left(L_{\varepsilon}\left(f_{\varepsilon}\right)\right)_{\varepsilon}$. According to lemma 27, we have for all $\varepsilon\in(0,1]$,

$$\begin{aligned}\widetilde{H}_{\varepsilon}\left(f_{\varepsilon}\right)-L_{\varepsilon}\left(f_{\varepsilon}\right)&=\left(L_{\varepsilon}\left(\psi_{\varepsilon}*f_{\varepsilon}\right)*\varphi_{\varepsilon}\right)-L_{\varepsilon}\left(f_{\varepsilon}\right),\\&=L_{\varepsilon}\left(\psi_{\varepsilon}*f_{\varepsilon}\right)*\varphi_{\varepsilon}-L_{\varepsilon}\left(f_{\varepsilon}\right)*\varphi_{\varepsilon}+L_{\varepsilon}\left(f_{\varepsilon}\right)*\varphi_{\varepsilon}-L_{\varepsilon}\left(f_{\varepsilon}\right),\\&=L_{\varepsilon}\left(\psi_{\varepsilon}*f_{\varepsilon}-f_{\varepsilon}\right)*\varphi_{\varepsilon}+L_{\varepsilon}\left(f_{\varepsilon}\right)*\varphi_{\varepsilon}-L_{\varepsilon}\left(f_{\varepsilon}\right).\end{aligned}$$

Remarking that $\left(f_{\varepsilon}\right)_{\varepsilon}\in\mathcal{F}_{\mathcal{L}_0}\left(\mathrm{C}^{\infty}(\Omega)\right)$ and $\left(L_{\varepsilon}\left(f_{\varepsilon}\right)\right)_{\varepsilon}\in\mathcal{F}_{\mathcal{L}_1}\left(\mathrm{C}^{\infty}(\Omega)\right)$ we get $\left(L_{\varepsilon}\left(f_{\varepsilon}\right)*\varphi_{\varepsilon}-L_{\varepsilon}\left(f_{\varepsilon}\right)\right)_{\varepsilon}\in\mathcal{N}\left(\mathrm{C}^{\infty}\left(\mathbb{R}^m\right)\right)$ and $\left(\psi_{\varepsilon}*f_{\varepsilon}-f_{\varepsilon}\right)_{\varepsilon}\in\mathcal{N}\left(\mathrm{C}^{\infty}\left(\mathbb{R}^m\right)\right)$ by lemma 16. This last property gives

$$\left(L_{\varepsilon}\left(\psi_{\varepsilon}*f_{\varepsilon}-f_{\varepsilon}\right)\right)_{\varepsilon}\in\mathcal{N}\left(\mathrm{C}^{\infty}\left(\mathbb{R}^m\right)\right)\quad\text{and}\quad\left(L_{\varepsilon}\left(\psi_{\varepsilon}*f_{\varepsilon}-f_{\varepsilon}\right)*\varphi_{\varepsilon}\right)\in\mathcal{N}\left(\mathrm{C}^{\infty}\left(\mathbb{R}^m\right)\right),$$

since $\left(\eta_{\varepsilon}*\varphi_{\varepsilon}\right)_{\varepsilon}\in\mathcal{N}\left(\mathrm{C}^{\infty}\left(\mathbb{R}^m\right)\right)$ for all $\left(\eta_{\varepsilon}\right)_{\varepsilon}\in\mathcal{N}\left(\mathrm{C}^{\infty}\left(\mathbb{R}^m\right)\right)$. Finally

$$\left[\left(\widetilde{H}_{\varepsilon}\left(f_{\varepsilon}\right)\right)_{\varepsilon}\right]=\left[\left(L_{\varepsilon}\left(f_{\varepsilon}\right)\right)_{\varepsilon}\right]=L\left(\left[\left(f_{\varepsilon}\right)_{\varepsilon}\right]\right),$$

this last equality by definition of the extension of a linear map.

5.2 Proof of proposition 23

Assertion i. We have only to prove continuity on 0. Let us fix $\varepsilon\in(0,1]$. Take $\left(f_k\right)_k\in\mathcal{D}\left(\mathbb{R}^n\right)^{\mathbb{N}}$ a sequence converging to 0 in $\mathcal{D}\left(\mathbb{R}^n\right)$. Since Λ is continuous, the sequence $\left(T_k\right)_k=\left(\Lambda\left(f_k\right)\right)_k$ tends to 0 in $\mathcal{D}'\left(\mathbb{R}^m\right)$ for the strong topology. Let us recall that [17]:

Lemma 28 *A sequence $\left(T_k\right)_k$ tends to 0 in $\mathcal{D}'\left(\mathbb{R}^m\right)$ for the strong topology if, and only if for all $\theta\in\mathcal{D}\left(\mathbb{R}^m\right)$ the sequence $\left(T_k*\theta\right)_k$ tends to 0, uniformly on every compact set.*

For all α in $\mathbb{N}^m$, we take $\theta_{\alpha}=\partial^{\alpha}\varphi_{\sqrt{\varepsilon}}$. Applying lemma 28, the sequences

$$\left(T_k*\partial^{\alpha}\varphi_{\sqrt{\varepsilon}}\right)_k=\left(\partial^{\alpha}\left(T_k*\varphi_{\sqrt{\varepsilon}}\right)\right)_k$$

tends to 0 uniformly on each compact of $\mathbb{R}^m$. Then L_{ε} is continuous.

Assertion ii. According to definition 19, we have to show that, for all $J\in\mathbb{N}$, the net $\left(L_{\varepsilon|\mathcal{D}_J}\right)_{\varepsilon}\in\left(\mathcal{L}\left(\mathcal{D}_J\left(\mathbb{R}^n\right),\mathcal{D}'\left(\mathbb{R}^m\right)\right)\right)^{(0,1]}$ is strongly moderate. We have

$$\begin{aligned}\forall f\in\mathcal{D}_J\left(\mathbb{R}^n\right),\ \forall x\in\mathbb{R}^m,\ \forall\alpha\in\mathbb{N}^m,\ \ \partial^{\alpha}\left(L_{\varepsilon\mathcal{D}_J|}(f)\right)(x)&=\left(\Lambda(f)*\partial^{\alpha}\varphi_{\sqrt{\varepsilon}}\right)(x),\\&=\left\langle\Lambda(f),\left\{y\mapsto\partial^{\alpha}\varphi_{\sqrt{\varepsilon}}(x-y)\right\}\right\rangle.\end{aligned}$$

Consider K a compact subset of $\mathbb{R}^m$. As $\operatorname{supp}\varphi_{\sqrt{\varepsilon}}$ decrease to $\{0\}$ for $\varepsilon \to 0$, there exists a compact K' such that

$$\forall x \in K,\ \ \forall \varepsilon \in (0,1]\,, \qquad \operatorname{supp}\left(\partial^\alpha \left(y \mapsto \varphi_{\sqrt{\varepsilon}}\,(x-y)\right)\right) \subset K'.$$

The map

$$\Theta : \mathcal{D}_J\left(\mathbb{R}^n\right) \times \mathcal{D}_{K'}\left(\mathbb{R}^m\right), \qquad (f,\varphi) \to \langle \Lambda\,(f)\,, \varphi\,(x-\cdot)\rangle$$

is a bilinear map separately continuous since Λ is continuous. As $\mathcal{D}_J\left(\mathbb{R}^n\right)$ and $\mathcal{D}_{K'}\left(\mathbb{R}^m\right)$ are Frechet spaces, Θ is globally continuous. There exists $C > 0$, $l_1 \in \mathbb{N}$, $l_2 \in \mathbb{N}$ such that

$$\forall\,(f,\varphi) \in \mathcal{D}_J\left(\mathbb{R}^n\right) \times \mathcal{D}_{K'}\left(\mathbb{R}^m\right), \quad \left|\langle \Lambda\,(f)\,, \varphi\rangle\right| \le C P_{J,l_1}(f) P_{K',l_2}(\varphi\,(x-\cdot)).$$

In particular, for $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^m$ with $|\alpha| \le l$, we have

$$\left|\langle \Lambda\,(f)\,, \partial^\alpha \varphi_\varepsilon\,(x-\cdot)\rangle\right| \le C P_{J,l_1}(f) P_{K',l_2}(\partial^\alpha \varphi_{\sqrt{\varepsilon}}\,(x-\cdot)), \tag{11}$$

and $P_{K',l_2}(\partial^\alpha \varphi_{\sqrt{\varepsilon}}\,(x-\cdot)) \le P_{K',l_2+l}(\partial^\alpha \varphi_{\sqrt{\varepsilon}}\,(x-\cdot))$.

Let us recall that

$$\partial^\alpha \varphi_{\sqrt{\varepsilon}}\,(x-\cdot) = \partial^\alpha \left\{ y \mapsto \left(\sqrt{\varepsilon}\right)^{-m} \varphi\left((x-y)/\sqrt{\varepsilon}\right) \kappa\left(\left|\ln \varepsilon\right|(x-y)\right)\right\}.$$

By induction on $|\alpha|$ and using the boundeness of φ, κ and their derivatives on $\mathbb{R}^m$, we can show that there exists a constant C_1, depending on $|\alpha|$, φ and κ and their derivatives but not on ε, such that

$$\sup_{y \in K'} \left|\partial^\alpha \left\{ y \mapsto \varphi_{\sqrt{\varepsilon}}\,(x-y)\right\}\right| \le C_1' \left(\sqrt{\varepsilon}\right)^{-(m+|\alpha|+1)}.$$

It follows that there exists a constant C_2 (independent of ε) such that

$$P_{K',l_2+l}(\varphi_\varepsilon\,(x-\cdot)) \le C_2 \left(\sqrt{\varepsilon}\right)^{-(m+l_2+l+1)}.$$

Putting this result in relation (11), we finally get the existence of a constant C_3 (independent of ε) such that

$$p_{K,l}(L_{\varepsilon \mathcal{D}_J|}(f)) = \sup_{x \in K,\ |\alpha| \le l} \left|\langle L\,(f)\,, \partial^\alpha \varphi_\varepsilon\,(x-\cdot)\rangle\right| \le C_3 \varepsilon^{-\frac{m+l_2+l+1}{2}} P_{J,l_1}(f).$$

The sequence $r\,(\cdot) = \left\{ l \mapsto \frac{m+l_2+l+1}{2} \right\}$ satisfies $\lim_{l \to +\infty} (r(l)/l) = 1/2$ showing our claim.$\blacksquare$

5.3 Proof of proposition 24

We first have the following:

Lemma 29 *For all* $T \in \mathcal{D}'\left(\mathbb{R}^m\right)$ $\left[\left(T * \varphi_{\sqrt{\varepsilon}}\right)_\varepsilon\right]$ *is equal to* T *in the generalized distribution sense.*

Proof. Take $T \in \mathcal{D}'\left(\mathbb{R}^m\right)$ and $g \in \mathcal{D}\left(\mathbb{R}^m\right)$, with $K = \operatorname{supp} g$. Set, for $\varepsilon \in (0,1]$,

$$A_{\sqrt{\varepsilon}} = \int_K \left(T * \varphi_{\sqrt{\varepsilon}}\right)(x)\, g\,(x)\ \mathrm{d}x = \int_K \left\langle T, \varphi_{\sqrt{\varepsilon}}\,(x-\cdot)\right\rangle g\,(x)\ \mathrm{d}x.$$

As $\operatorname{supp}\varphi_{\sqrt{\varepsilon}}$ decrease to $\{0\}$ for $\varepsilon \to 0$, there exists a relatively compact open subset Ω such that

$$\forall x \in K,\ \ \forall \varepsilon \in (0,1]\,, \qquad \operatorname{supp}\left(y \mapsto \varphi_{\sqrt{\varepsilon}}\,(x-y)\right) \subset \Omega.$$

There exists f continuous with compact support and $\alpha \in \mathbb{N}^m$ such that $T_{|\Omega} = \partial^\alpha f$. This implies that $\left\langle T, \varphi_{\sqrt{\varepsilon}}(x-\cdot)\right\rangle = \left\langle \partial^\alpha f, \varphi_{\sqrt{\varepsilon}}(x-\cdot)\right\rangle$ and

$$\left(T * \varphi_{\sqrt{\varepsilon}}\right)(x) = \left(\partial^\alpha f * \varphi_{\sqrt{\varepsilon}}\right)(x) = \partial^\alpha \left(f * \varphi_{\sqrt{\varepsilon}}\right)(x).$$

By integration by part (g is compactly supported) it follows that

$$A_{\sqrt{\varepsilon}} = \int_K \partial^\alpha \left(f * \varphi_{\sqrt{\varepsilon}}\right)(x)\, g(x)\, \mathrm{d}x = (-1)^{|\alpha|} \int_K \left(f * \varphi_{\sqrt{\varepsilon}}\right)(x)\, \partial^\alpha g(x)\, \mathrm{d}x.$$

Consider now an integer k and $\beta \in \mathbb{N}^m$ such that $\beta = \beta_1 + \ldots + \beta_m$ with $\beta_j \geq k$, for each $j \in \{1, \ldots, m\}$. We consider F_β a function such that $\partial^\beta F_\beta = f$, which exists since f is continuous. This function is at least of class C^k. We have

$$\begin{aligned}
A_{\sqrt{\varepsilon}} &= (-1)^{|\alpha|} \int_K \left(\partial^\beta F_\beta * \varphi_{\sqrt{\varepsilon}}\right)(x)\, \partial^\alpha g(x)\, \mathrm{d}x, \\
&= (-1)^{|\alpha|+|\beta|} \int_K \left(F_\beta * \varphi_{\sqrt{\varepsilon}}\right)(x)\, \partial^{\alpha+\beta} g(x)\, \mathrm{d}x, \\
\langle T, g\rangle = \langle \partial^\alpha f, g\rangle &= \left\langle \partial^{\alpha+\beta} F_\beta, g\right\rangle = (-1)^{|\alpha|+|\beta|} \left\langle F_\beta, \partial^{\alpha+\beta} g\right\rangle, \\
&= (-1)^{|\alpha|+|\beta|} \int_K (F_\beta)(x)\, \partial^{\alpha+\beta} g(x)\, \mathrm{d}x.
\end{aligned}$$

Then

$$\left\langle T * \varphi_{\sqrt{\varepsilon}}, g\right\rangle - \langle T, g\rangle = (-1)^{|\alpha|+|\beta|} \int_K \left(\left(F_\beta * \varphi_{\sqrt{\varepsilon}}\right)(x) - (F_\beta)(x)\right) \partial^{\alpha+\beta} g(x)\, \mathrm{d}x.$$

An adaptation (and simplification) of the proof of lemma 16 shows that

$$\left(F_\beta * \varphi_{\sqrt{\varepsilon}}\right)(x) - (F_\beta)(x) = \mathrm{O}\left(\sqrt{\varepsilon}^k\right) \text{ for } \varepsilon \to 0.$$

As g is compactly supported, this last relation leads to

$$\left\langle T * \varphi_{\sqrt{\varepsilon}}, g\right\rangle - \langle T, g\rangle = \mathrm{O}\left(\sqrt{\varepsilon}^k\right) \text{ for } \varepsilon \to 0.$$

Since k is arbitrary, our claim follows. ■

This lemma implies that for all $f \in \mathcal{D}(\mathbb{R}^n)$, $[(L_\varepsilon(f))_\varepsilon] = \left[\left(\Lambda(f) * \varphi_{\sqrt{\varepsilon}}\right)_\varepsilon\right]$ is equal to $\Lambda(f)$ in the generalized distribution sense. On the other hand, according to theorem 21, $[(L_\varepsilon(f))_\varepsilon] = \widetilde{H}_L(f)$ where $\widetilde{H}_L$ is the integral operator associated to the canonical extension of $(L_\varepsilon)_\varepsilon$. This ends the proof.

References

[1] S. Bernard, J.-F. Colombeau, A. Delcroix. "Generalized Integral Operators". Preprint AOC, 2003.

[2] J.-F. Colombeau. *New Generalized Functions and Multiplication of Distributions.* North-Holland, Amsterdam, Oxford, New-York, 1984.

[3] A. Delcroix, M. Hasler, S. Pilipović, V. Valmorin. "Generalized function algebras as sequence space algebras". To appear in Am. Math. Proc.

[4] A. Delcroix, M. Hasler, S. Pilipović, V. Valmorin. “Algebras of generalized functions through sequence spaces algebras. Functoriality and associations”. International Journal of Mathematical Sciences **1-2**:13-31, 2002.

[5] A. Delcroix D. Scarpalezos. “Topology on Asymptotic Algebras of Generalized Functions and Applications”. Mh. Math. **129**:1-14, 2000.

[6] S. Garetto, T. Gramchev, M. Oberguggenberger. “Pseudo-Differential operators and regularity theory” Preprint, 2003.

[7] M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer. *Geometric Theory of Generalized Functions with Applications to General Relativity.* Kluwer Academic Press, 2001.

[8] X. Grothendieck. *Produits tensoriels topologiques et espaces nucléaires.* Memoirs of the Am. Math. Soc. 16, 1955.

[9] L. Hormander. *The analysis of Linear Partial Differential Operators I, distribution theory and Fourier Analysis.* Grundlehren der mathematischen Wissenchaften 256. Springer Verlag, Berlin, Heidelberg, New York, 2nd edition, 1990.

[10] J.-A. Marti. “Fundamental structures and asymptotic microlocalization in sheaves of generalized functions”. Integral Transforms and Special Functions **6**(1-4):223-228, 1998.

[11] J.-A. Marti. “Non linear Algebraic analysis of delta shock wave to Burgers’ equation”. Pacific Journal of Mathematics **210**(1):165-187, 2003.

[12] M. Oberguggenberger. *Multiplication of Distributions and Applications to Partial Differential Equations.* Longman Scientific & Technical, 1992.

[13] M. Oberguggenberger, S. Pilipović, D. Scarpalezos. “Local properties of Colombeau generalized functions”. Math. Nachr. **256**:88-99, 2003.

[14] S. Pilipović, D. Scarpalezos. “Colombeau generalized ultradistributions”. Math. Proc. Camb. Phil. Soc. **130**:541-553, 2001.

[15] M. Nedeljkov, S. Pilipović, D. Scarpalezos. *The linear theory of Colombeau generalized functions.* Pitman Research Notes in Mathematics Series 385. Longman, 1998

[16] D. Scarpalezos. *Colombeau’s generalized functions: Topological structures; Microlocal properties. A simplified point of view.* Prépublication Mathématiques de Paris 7/CNRS, URA212, 1993.

[17] L. Schwartz. *Théorie des Distributions.* Hermann, 1966.

[18] L. Schwartz. *Distributions à valeurs vectorielles.* Ann. Inst. Fourier **7**:1-141, 1957 and **8**:1-209, 1959.